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New Results on Finite Polarized Partition Relations

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Let $c = c(m, n, j, k)$ be the largest integer such that every matrix with m rows and n columns whose entries belong to a set of cardinal c has a constant submatrix with j rows and k columns. Some results in the case $j = 2$ are given.

1. INTRODUCTION

In 1951 Zarankiewicz [8] introduced the following problem: determine the smallest integer $z = z(m, n, j, k)$ such that every 0–1 matrix with m rows and n columns containing z ones has a submatrix with j rows and k columns entirely filled with ones.

The number $z(m, n, j, k) - 1$ can equivalently be defined as the weak stability number of the hypergraph $K_m^j \times K_n^k$ (direct product of the complete j-uniform hypergraph of order m by the complete k-uniform hypergraph of order n).

In 1956, Erdös and Rado [3] introduced the following type of problem: determine the largest integer $c = c(m, n, j, k)$ such that every matrix with m rows and n columns whose entries belong to a set of cardinal c has a constant submatrix with j rows and k columns. They called this kind of properties "finite polarized partition relations." References on this problem are [1–4, 7].

The number $c(m, n, j, k) + 1$ can equivalently be defined as the weak chromatic number of $K_m^j \times K_n^k$. Hence Zarankiewicz's problem and the Erdös–Rado problem are closely related; indeed

LEMMA 1.1. $(z(m, n, j, k) - 1)(c(m, n, j, k) + 1) \geqslant mn$.

Proof. By definition, there exists an $m \times n$ matrix M whose mn entries belong to a set of symbols of cardinal $c(m, n, j, k) + 1$ with no constant $j \times k$ submatrix. By the definition of $z(m, n, j, k)$, each of these symbols occurs at most $z(m, n, j, k) - 1$ times as an entry of M. Hence, $(z(m, n, j, k) - 1)$ $(c(m, n, j, k) + 1) \geqslant mn$.

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Remark. Lemma 1.1 is a particular case of the classical property of hypergraphs: the product of the weak stability number by the weak chromatic number is greater than or equal to the number of vertices.

The following results on the Erdös–Rado problem are based on an idea of Mörs [6] for Zarankiewicz's problem.

In the sequel we assume $k \geqslant 2$.

2. Main Result

THEOREM 2.1. *Let p be a prime number such that $q = (p-1)/(k-1)$ is an integer. Then, $c(pq, pq, 2, k) \leqslant q + \lceil q/(k-1) \rceil - 1$.*

Proof. It is sufficient to prove that there exists a $pq \times pq$ matrix M filled with $q + \lceil q/(k-1) \rceil$ symbols containing no constant $2 \times k$ submatrix. Let $Q = \{1, 2, ..., q\}$.

Let $R = \{(a, b) \mid a \in \mathbb{Z}_p, b \in Q\}$ be the index set for the rows of M, and let $C = \{(e, f) \mid e \in \mathbb{Z}_p, f \in Q\}$ the index set for the columns of M.

Let r be a primitive root of the field $\mathbb{Z}_p$, and G be the multiplicative subgroup of $\mathbb{Z}_p - \{0\}$ generated by r^q, so that $|G| = k-1$.

Let us define the matrix M as follows:

(1) Entries $M((a, b), (e, f))$ such that $e = r^f a$.

Let $S = \{s_1, ..., s_{\lceil q/(k-1) \rceil}\}$ be a set of symbols. Then, we take

$$M((a, b), (r^f a, f)) = s_u \qquad \text{with} \quad u = \lceil f/(k-1) \rceil.$$

Since every symbol s_u occurs at most $k-1$ times in any given row, there is no $2 \times k$ submatrix entirely filled with only one symbol s_u.

(2) Entries $M((a, b), (e, f))$ such that $e \neq r^f a$.

Let $T = \{t_0, ..., t_{q-1}\}$ be a set of symbols. If a, b, e, and f are given, let $h \in \{0, ..., q-1\}$ and $i \in \mathbb{Z}_{k-1}$ be uniquely defined by $e - r^f a = r^{b+h+iq}$ (r^h represents $(e - r^f a)/r^b$ modulo G; we may take i in the set $\mathbb{Z}_{k-1}$ since $r^{(k-1)q} = 1$). Then, we take $M((a, b), (e, f)) = t_h$.

We have to prove that there exists no $2 \times k$ submatrix entirely filled with only one of the symbols t_h. Let us suppose, indirectly, that such a submatrix exists: let (a, b) and (a', b'), with $(a, b) \neq (a', b')$, be its row indices. If (e, f) is a column index of this submatrix, then $M((a, b), (e, f)) = M((a', b'), (e, f)) = t_h$. Hence, $e - r^f a = r^{b+h+iq}$ and $e - r^f a' = r^{b'+h+jq}$ for some $i \in \mathbb{Z}_{k-1}$ and some $j \in \mathbb{Z}_{k-1}$. Hence we have the equation in $\mathbb{Z}_p$

$$r^f a + r^{b+h+iq} = r^f a' + r^{b'+h+jq}. \tag{2.1}$$

Hence, to every column index (e, f) of the submatrix corresponds a triple

(f, i, j) in $Q \times \mathbb{Z}_{k-1} \times \mathbb{Z}_{k-1}$ which satisfies (2.1). Since the value of e can be obtained from f and i, this mapping is one-to-one.

Hence, there exist k triples (f, i, j) in $Q \times \mathbb{Z}_{k-1} \times \mathbb{Z}_{k-1}$ which satisfy (2.1). But this is impossible by Lemma 2.3 (see proof). Indeed, for every value of m in Lemma 2.3, there is at most one triple (f, i, j) which satisfies (2.1), and m can take only $k-1$ values.

As in [6], Lemma 2.3 will be proved by using a preliminary result:

LEMMA 2.2. *For fixed $(i, j) \in \mathbb{Z}_{k-1} \times \mathbb{Z}_{k-1}$, there exists at most one integer $f \in \{1, 2, ..., p-1\}$ such that (f, i, j) satisfies* (2.1).

Note that f ranges here over a set larger than Q.

Proof. Suppose indirectly that there exists $f \neq f' \in \{1, 2, ..., p-1\}$ such that

$$r^f a + r^{b+h+iq} = r^f a' + r^{b'+h+jq}$$
$$r^{f'} a + r^{b+h+iq} = r^{f'} a' + r^{b'+h+jq}.$$

Then $(r^f - r^{f'})(a - a') = 0$. Hence $a = a'$, $b = b'$, and $i = j$: a contradiction, since $(a, b) \neq (a', b')$.

LEMMA 2.3. *For fixed $m = j - i \in \mathbb{Z}_{k-1}$, there exists at most one triple (f, i, j) in $Q \times \mathbb{Z}_{k-1} \times \mathbb{Z}_{k-1}$ which satisfies* (2.1).

Proof. Suppose indirectly that $(f, i, i+m)$ and $(f', i', i'+m) \in Q \times \mathbb{Z}_{k-1} \times \mathbb{Z}_{k-1}$ satisfy (2.1), with $i \neq i'$ (by Lemma 2.2). Then,

$$\begin{aligned} r^f a + r^{b+h+iq} &= r^f a' + r^{b'+h+iq+mq} \\ r^{f'} a + r^{b+h+i'q} &= r^{f'} a' + r^{b'+h+i'q+mq}. \end{aligned} \tag{2.2}$$

Let $n \in \{1, 2, ..., k-2\}$ be such that the class of n modulo $k-1$ is equal to $i - i'$. Then, multiplying (2.2) by r^{nq}

$$r^{f'+nq} a + r^{b+h+iq} = r^{f'+nq} a' + r^{b'+h+iq+mq}.$$

Then $(f, i, i+m)$ and $(f'+nq, i, i+m)$ with $f \neq f'+nq \in \{1, 2, ..., p-1\}$, would satisfy (2.1), contrary to Lemma 2.2.

THEOREM 2.4. *Let p be an integer such that $q = (p-1)/(k-1)$ is an integer. Then, $c(pq, pq, 2, k) \geqslant q$.*

Proof. Hylten-Cavallius [5] proved the formula

$$z(m, n, 2, k) - 1 \leqslant (n + (n^2 + 4n(k-1)\, m(m-1))^{1/2})/2.$$

Hence, $z(pq, pq, 2, k) - 1 \leqslant pq(1 + (1 + 4(k-1)(pq-1))^{1/2})/2$. This inequality, together with Lemma 1.1 yields

$$c(pq, pq, 2, k) + 1 \geqslant 2pq/(1 + (1 + 4p(p-1) - 4(k-1))^{1/2}).$$

Hence $c(pq, pq, 2, k) + 1 > q$, and $c(pq, pq, 2, k) + 1 \geqslant q + 1$.

THEOREM 2.5. *Let p be a prime such that $q = (p-1)/(k-1)$ is an integer not exceeding $k-1$. Then, $c(pq, pq, 2, k) = q$.*

Proof. This follows immediately from Theorems 2.4 and 2.1.

Theorem 2.5 gives, for example, the following exact values: $c(10, 10, 2, 3) = 2$, $c(14, 14, 2, 4) = 2$, $c(39, 39, 2, 5) = 3$, $c(68, 68, 2, 5) = 4$, etc.

3. ASYMPTOTIC RESULTS

Mörs proved that $z(n, n, 2, k)/n^{3/2} \to (k-1)^{1/2}$ when $n \to \infty$. Let k be a fixed integer. It is known that for every real $\varepsilon > 0$, and every integer n large enough, there exists a prime number p such that

$$(1-\varepsilon)n \leqslant p \leqslant n \qquad \text{and} \qquad k-1 \quad \text{divides} \quad p-1. \tag{3.1}$$

From this remark and Theorem 2.1 it follows that

$$\limsup_{n \to \infty} c(n, n, 2, k)/n^{1/2} \leqslant (k-1)^{-1/2} + (k-1)^{-3/2}. \tag{3.2}$$

On the other hand, Theorem 2.4 implies

$$\liminf_{n \to \infty} c(n, n, 2, k)/n^{1/2} \geqslant (k-1)^{-1/2}. \tag{3.3}$$

In order to get more precise asymptotic results we now improve Theorem 2.1.

THEOREM 3.1. *Let p be a prime number such that $q = (p-1)/(k-1)$ is an integer. Then $c(pq, pq, 2, k) \leqslant q + c(q, q, 2, k)$.*

Proof. The proof goes as in Theorem 2.1 except for part (1). If a is fixed, the entries $M((a, b), (r^f a, f))$ form a $q \times q$ submatrix N_a. By definition, it is possible to fill N_a with symbols from a set S of cardinal $c(q, q, 2, k) + 1$ without having a constant $2 \times k$ submatrix. The same set S may be used for all matrices N_a, $a \in \mathbb{Z}_p$, since their row and column index sets are pairwise disjoint.

THEOREM 3.2. $\lim_{n\to\infty} c(n, n, 2, k)/n^{1/2} = (k-1)^{-1/2}$.

Proof. This follows from (3.1)–(3.3) and Theorem 3.1.

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